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ON APPLICATION OF GENERALIZED MITTAG-LEFFLER TYPE FUNCTION

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ABSTRACT
In this paper, authors attempt to derive certain results based on the generalized Mittag-Leffler type functions namely generalized M-Series earlier introduced by Sharma et al. [8] in context to statistical distribution theory. Special cases of interest have also been discussed.

KEYWORDS: Generalized Mittag-Leffler function, generalized M-series, Statistical distribution theory.

INTRODUCTION

Mittag-Leffler functions occur naturally in the solution of fractional order differential and integral equations [5]. The Mittag-Leffer (ML) function has earned a universal recognition of being the Queen-function among the class of SFs owing to its adaptability. The various Mittag-Leffler type functions ([1], [2], [4], [7], [8], [9], [12]) discussed in this paper will be useful for investigators in various disciplines of applied sciences and engineering. The importance of such functions in physics is steadily increasing. It is simply said that deviations of physical phenomena from exponential behavior could be governed by physical laws through Mittag-Leffler functions (power law). Currently more and more such phenomena are discovered and studied. It is particularly important for the disciplines of stochastic systems, dynamical systems theory and disordered systems. Eventually, it is believed that all these new research results will lead to the discovery of truly non-equilibrium statistical mechanics. This is statistical mechanics beyond Boltzmann and Gibbs. This non-equilibrium statistical mechanics will focus on entropy production, reaction, diffusion, reaction-diffusion and so forth, and may be governed by fractional calculus. Right now, fractional calculus and generalization of Mittag-Leffler functions are very important in research in physics. These presentations make the reader familiar with the present trend of research in Mittag-Leffler type functions and their applications. Special functions have contributed a lot to mathematical physics and its various branches. The great use of mathematical physics in distinguished astrophysical problems has attracted astronomers and physicists to pay more attention to available mathematical tools that can be widely used in solving several problems of astrophysics/physics. Throughout this paper, we need the following definitions

The Swedish mathematician Mittag-Leffler [4] introduced the function $E_{\sim}(x)$ defined as

$$E_{\sim}(x)=\sum_{n=0}^{\infty}\frac{x^{n}}{\Gamma(\sim n+1)},\ \ (x,\sim\in C, R(\sim)>0) \quad \textbf{......(1)}$$

A generalization of $E_{\sim}(x)$ was studied by Wiman[2] where he defined the function $E_{\sim,\ }(x)$ as

$$E_{\sim,\ }(x)=\sum_{n=0}^{\infty}\frac{x^{n}}{\Gamma(\sim n+\ \ )},$$

$$(x,\sim,\ \ \in C, R(\sim)>0, R(\ \ )>0) \quad ....(2)$$

which is also known as Mittag-Leffler function or Wiman's function.

A generalization of (1) and (2) was introduced by Prabhakar [11] in terms of the series representation

$$E_{\sim,\ }^{¥}(x)=\sum_{r=0}^{\infty}\frac{(¥)_{n}x^{r}}{r!\Gamma(\sim r+\ \ )} \quad ....(3)$$

where $\sim,\ \ ,¥\in C, \mathrm{Re}(\sim), \mathrm{Re}(\ \ )>0\ (¥)_{n}$ is Pochammer's symbol defined by

$$(¥)_{n}=¥(¥+1).......((¥+(n-1)), n\in N, ¥\neq 0.$$

It is an entire function of order $° = [\mathrm{Re}(\sim)]^{-1}.$

A generalization of (3) was defined by Sharma [9] as

$${}_{p}\tilde{M}_{q}(a_{1},\ldots,a_{p;}b_{1},\ldots,b_{q};x)=$$

$${}_{p}\tilde{M}_{q}(x)=\sum_{r=0}^{\infty}\frac{(a_{1})_{r}\ldots(a_{p})_{r}}{(b_{1})_{r}\ldots(b_{q})_{r}}\frac{x^{r}}{\Gamma(\sim r+1)} \quad ....(4)$$

where $\sim \in C, \mathrm{Re}(\sim) > 0$ and $(a_j)_r$ and $(b_j)_r$ are the Pochammer symbols. The detailed information of this series is given in [9].
A generalization of (4) was given by Sharma and Jain [8] as

$$ {}_pM_q^{\sim,}\ (a_1,\ldots,a_p; b_1,\ldots,b_q; x) = $$

$$ {}_pM_q^{\sim,}\ (x) = \sum_{r=0}^{\infty} \frac{(a_1)_r\ldots(a_p)_r}{(b_1)_r\ldots(b_q)_r} \frac{x^r}{\Gamma(\sim r + \ \ )} \quad \ldots\ldots\ldots\ldots(5) $$

here $\sim, \ \ \in C, \mathrm{Re}(\sim) > 0$ and $(a_j)_r$ and $(b_j)_r$ are the Pochammer symbols. The detailed information of this series is given in [8].
It is easily seen that equation (5) is an obvious generalization of equations (1) to (4) and exponential function as follows

(i) If we take $\ \ = 1$, we arrive at

$$ {}_pM_q^{\sim,1}(x) = {}_pM_q^{\sim}(x) \quad \ldots\ldots\ldots\ldots\ldots(6) $$

where ${}_pM_q^{\sim}(x)$ is the M-series given by Sharma[9].

(ii) If we put $p = q = 0$, we arrive at

$$ {}_0M_0^{\sim,}\ (x) = E_{\sim,}\ (x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(7) $$

where $E_{\sim,}\ (x)$ is the generalized Mittag-Leffler function given by Wiman [2].

(iii) If we put $p = q = 0, \ \ = 1$, we arrive at

$$ {}_0M_0^{\sim,1}(x) = E_{\sim,1}(x) = E_{\sim}(x) \quad \ldots\ldots\ldots\ldots\ldots(8) $$

where $E_{\sim}(x)$ is the Mittag-Leffler function[4].

(iv) If we put $p = q = 0, \sim = \ \ = 1$, we get

$$ {}_0M_0^{1,1}(x) = E_{1,1}(x) = E_1(x) = e^x \quad \ldots\ldots\ldots\ldots\ldots\ldots(9) $$

where e^x is the exponential function given by Rainville [3].

Distribution Function [14]

Let X be a random variable and let $x_1, x_2, \ldots$ be the values which it assumes; in most of what follows the x_j will be integer. The aggregate of all sample points on which X assumes the fixed value x_j forms the event $X = x_j$; its probability is defined by $P\{X = x_j\}$. The function $P\{X = x_j\} = f(x_j), j = 1,2,\ldots$ is called the probability distribution of the random variable. Clearly, $f(x_j) \geq 0. \sum f(x_j) = 1.$

The distribution function $F(x)$ of X is defined by

$$ F(x) = P\{X \leq x\} = \sum_{x_j \leq x} f(x_j). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10) $$

The last sum extending over all those x_j which do not exceed x. $F(x)$ is a non-decreasing function which tends to zero as $x \to -\infty$ and to one as $x \to \infty$. Thus the distribution function can be calculated from its probability distribution and vice versa.
Bernoulli number, digamma function and extended Bernoulli number are respectively given by [11],[6] and [10].

The generalized M-Series in Statistical Distribution

Theorem 2.1 If $G_y(y) = 1 - {}_pM_q^{\sim,1}(-y^{\sim})$ then

$$ f(y) = y^{\sim -1}\, {}_pM_q^{\sim,\sim}(-y^{\sim}); 0 < \sim \leq 1, y > 0. \quad \ldots\ldots\ldots\ldots(11) $$

Proof:
We have

$$ G_y(y) = 1 - {}_pM_q^{\sim,1}(-y^{\sim}) $$

From the definition (5), we get

$$ = 1 - \sum_{n=0}^{\infty} \frac{(a_1)_n\ldots(a_p)_n}{(b_1)_n\ldots(b_q)_n} \frac{(-y^{\sim})^n}{\Gamma(\sim n + 1)} $$

On simplifying

$$ = \sum_{n=0}^{\infty} \frac{(a_1)_n\ldots(a_p)_n}{(b_1)_n\ldots(b_q)_n} \frac{(-1)^{n+1} y^{\sim n}}{\Gamma(\sim n + 1)} \quad \ldots\ldots\ldots\ldots\ldots(12) $$

The density function is given by

$$ f(y) = \frac{d}{dy}[\ G_y(y)] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(13) $$

Using (12) in (13), we obtain

$$ = \sum_{n=1}^{\infty} \frac{(a_1)_n\ldots(a_p)_n}{(b_1)_n\ldots(b_q)_n} \frac{(-1)^{n+1} \sim n y^{\sim n - 1}}{\Gamma(\sim n + 1)} $$

On replacing $n \to n + 1$

$$ = \sum_{n=0}^{\infty} \frac{(a_1)_n\ldots(a_p)_n}{(b_1)_n\ldots(b_q)_n} \frac{(-1)^{n+1}(\sim n + \sim) y^{(\sim n + \sim) - 1}}{\Gamma((\sim n + \sim) + 1)} \quad \ldots\ldots.(14) $$

In accordance with (6), we arrive at the desired result.

Special Cases:
From Theorem 2.1, we can easily obtain the following results.

If we take $p = q = 0$ in equation (14), which reduces to the well known result obtained by Haubold et al. [5]:

Corollary 2.1

If $G_y(y) = 1 - E_{\sim,1}(-y^{\sim})$ then

$$f(y) = y^{\sim-1} E_{\sim,\sim}(-y^{\sim}); 0 < \sim \leq 1, y > 0. \quad ....(15)$$

where $E_{\sim,\sim}(.)$ is the generalized Mittag-Leffler function introduced by Wiman[2].

If we take $p = q = 0, \sim = 1$ in equation (14), which reduces to result:

Corollary 2.2

If $G_y(y) = 1 - E_{1,1}(-y\ )$ then

$$f(y) = E_{1,1}(-y\ ); 0 < \sim \leq 1, y > 0. \quad ....(16)$$

where $E_{1,1}(-y\ ) = e^{-y}$ is the exponential function [3].

Remarks: It may be noted that theorem 2.1 is also an extension of the result derived earlier by Dhakar and Sharma [13].

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