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THE BIFURCATION OF THE DYNAMICS OF PREY-PREDATOR MODEL WITH HARVESTING INVOLVING DISEASES IN BOTH POPULATIONS

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ABSTRACT
In this paper, an eco-epidemiological system which considers a prey–predator system and (SI) disease with harvesting, using Holling type II as a functional response for the susceptible predator , linear functional response for the infected predator and the harvesting effect on the infectious population. Bifurcation such as (saddle node, transcritical and pitchfork) of the proposed system is investigated by using Sotomayrs theory and Hopf bifurcation theory; it's observed that there is transcritical bifurcation near axial equilibrium point ,the predator-free equilibrium point ,the disease-free equilibrium point ,the infected-predator-free equilibrium point and the infected-prey-free equilibrium point while there is a saddle–node bifurcation near coexistence equilibrium point, on the other hand there is no pitchfork bifurcation near all of these equilibrium points. Further investigations for the Hopf bifurcation near coexistence equilibrium point are carried out. Finally, numerical simulations are used to illustration the occurrence of local bifurcation of this system.

KEY WORDS: Eco-epidemiological Model, Local bifurcation, Sotomayor's theorem, Hopf bifurcation

INTRODUCTION

From the see of human needs, the utilization of biological resources and harvest of population are commonly practiced in prey-predator systems. But then, unreasonable exploitation of biological resources might lead to unfavorable influence on ecological balance. So there has been rapidly growing interest in the analysis and modeling of predator–prey systems. Many author[7–9] have studied the dynamics of prey-predator models with harvesting and disease, and obtained complex dynamic behaviors, such as Hopf bifurcation or periodic solution. However, they have not considered the effect of the harvest effort on ecosystem from an economic perspective or environmental[6]. In any case the continuous dynamical systems are usually composed from a set of the ordinary differential equations or a set of partial differential equations as well as a set of different parameters that control the nature of the system. The solution of these equations depends entirely on these parameters. These systems describe a problem in the medical, engineering, environmental or economic. Any simple or smooth change in any parameter present in the system may result in sudden change or topological change in its behavior, changing the nature of the system from stable to unstable or periodic or converse. Then this model (system) is said to be has a bifurcation. the bifurcation object is not exist only the subject of dynamical systems, it is found in various fields, for example, found in medicine, geometry, *etc*. Where the term was first introduced by the scientific Henri Poincaré Carré in 1885. The usefulness of bifurcation theory transcends our ability to cite theorems. By furnishing a qualitative modeling mechanism, it provides a conceptual framework within which we can view a number of important ecological processes. It is useful to divide bifurcations into two principal classes: local bifurcations, which can be analyzed entirely through changes in the local stability properties of equilibrium, periodic orbits or other invariant sets as parameters cross through critical thresholds and examples of local bifurcations include: saddle-node (fold) bifurcation, transcritical bifurcation, pitchfork bifurcation, period-doubling bifurcation and Hopf bifurcation; and global bifurcations, which often occur when larger invariant sets of the system with each other, or with equilibrium of the system. They cannot be detected purely by a stability analysis of the equilibria (fixed points). This causes changes in the topology of the trajectories in phase space which cannot be confined to a small neighborhood, as is the case with local bifurcations[6]. The term Hopf bifurcation (also sometimes called Poincar´e-Andronov-Hopf bifurcation) refers to exist or not exist a periodic solution from equilibrium as a parameter crosses a critical value. It is the simplest bifurcation not just involving equilibrium and therefore belongs to what is sometimes called dynamic bifurcation theory. In a differential equation a Hopf bifurcation typically occurs when a complex conjugate pair of eigenvalues of the linear flow at a fixed point becomes purely imaginary. This implies that a Hopf bifurcation can only occur in systems of dimension two or higher. The subject of the bifurcation and in particular Hopf is a very important subject in applied mathematics. Recently, Tayeh and Naji[5] had studied local bifurcation such as (saddle-node, transcritical and pitchfork) and Hopf bifurcation around each of the equilibrium points of prey predator model involving SI infection disease in both the prey and predator species and the disease transmitted by contact only. Khalaf ,Majeed and Naji[11] established the conditions of the occurrence of local bifurcation such as (saddle-node, transcritical and pitchfork) with particular emphasis on the Hopf bifurcation near the positive equilibrium point of prey-predator model involving SIS infectious disease in prey population this disease passed from a prey to predator through attacking of predator to prey and the disease transmitted within the same species by contact and external source.

In this paper, an application of Sotomayor's theorem [2,3] for local bifurcation is used to study the occurrence of local bifurcation near the equilibrium, furthermore the condition of occurrence of the Hopf bifurcation near positive equilibrium point are established of a mathematical model proposed by Majeed and Ali [4].

Model Formulation [4]

An eco-epidemiological mathematical model consisting of prey-predator model involving SI infectious disease with harvesting in infected population is proposed and analyzed in [4].

$$\begin{aligned}
\frac{ds}{dT} &= rs\left(1-\frac{s+I}{k}\right)-\beta_1 SI-\frac{a_1 SX}{b+S}\\
\frac{dI}{dT} &= \beta_1 SI - a_2 IX - a_3 IY - d_1 I - \square_1 I\\
\frac{dX}{dT} &= e_1\frac{a_1 SX}{b+S}+e_2 a_2 IX-\beta_2 XY-d_2 X\\
\frac{dY}{dT} &= \beta_2 XY + e_3 a_3 IY-(d_2+\alpha)Y-\square_2 Y.
\end{aligned} \qquad (2.1)$$

Where 0< e_i < 1; i = 1,2,3 represent the conversion rate constants and β_1 represents the infection rate of susceptible prey, β_2 represents the infection rate of susceptible predator. Note that, there is an SI epidemic disease in prey population divides the prey population into two classes namely S(T) that represents the density of susceptible prey species at time T and I(T) which represents the density of infected prey species at time T, and there is different disease divides the predator population into two classes namely X(T) that represents the density of susceptible predator species at time T and Y(t) that represents the density of infected predator species at time T . Therefore at any time T, we have N (T) = S (T) + I (T) and P(T)=X(T)+Y(T) , the diseases are not transmitted from prey to predator or converse , but it are transmitted in the same species all the parameters are moreover assumed to be positive and described as given in[4].

Now, for further simplification of the system (2.1), the following dimensionless variables are used in [4].

$$t = r\,T\,, x=\frac{S}{k}\ ,y=\frac{I}{k}\ ,z=\frac{X}{k}\,,w=\frac{Y}{k}.$$

Then system (2.1) can be written in the following dimensionless form:

$$\begin{aligned}
\frac{dx}{dt} &= x\left(1-x-y-c_1 y-\frac{c_2 z}{c_3+x}\right)=f_1(x,y,z,w)\\
\frac{dy}{dt} &= y\big(c_1 x-c_4 z-c_5 w-(c_6+c_7)\big)=f_2(x,y,z,w)\\
\frac{dz}{dt} &= z\left(\frac{c_8 x}{c_3+x}+c_9 y-c_{10} w-c_{11}\right)=f_3(x,y,z,w)\\
\frac{dw}{dt} &= w\big(c_{10} z+c_{12} y-(c_{11}+c_{13}+c_{14})\big)=f_4(x,y,z,w)
\end{aligned} \qquad (2.2)$$

where

$$c_1=\frac{\beta_1 k}{r}\,,c_2=\frac{a_1}{r}\,,c_3=\frac{b}{k}\,,c_4=\frac{a_2 k}{r}\,,c_5=\frac{a_3 k}{r}\,,c_6=\frac{d_1}{r}\,,c_7=\frac{\square_1}{r}\,,c_8=\frac{e_1 a_1}{r}\,,$$

$$c_9=\frac{e_2 a_2 k}{r}\,,c_{10}=\frac{\beta_2 k}{r}\,,c_{11}=\frac{d_2}{r}\,,c_{12}=\frac{e_3 a_3 k}{r}\,,c_{13}=\frac{\alpha}{r},c_{14}=\frac{\square_2}{r}.$$

With $x(0)\geq 0$,$y(0)\geq 0$, $z(0)\geq 0$ and $w(0)\geq 0$.

Represent the dimensionless parameter of system (2.2). It is observed that the number of parameters have been reduced from sixteen in the system (2.1) to fourteen in the system (2.2).

It is easy to verify that all the interaction functions f_1, f_2, f_3 and f_4 on the right hand side of system (2.2) are continuous and have continuous partial derivatives on R_+^4 with respect to dependent variables x , y , z and w. Accordingly they are Lipschitzian functions and hence system (2.2) has a unique solution for each non-negative initial condition. Further the boundedness of the system is shown in the following theorem.

Theorem (2.1)[4]: All the solutions of system (2.2) which initiate in R_+^4 are uniformly bounded.

Local bifurcation analysis:

In this section, the effect of varying the parameter values on the dynamical behavior of the system (2.2) around each equilibrium points is studied. Recall that the existence of non- hyperbolic equilibrium point of system (2.2) is the necessary but not sufficient condition for bifurcation to occur. Therefore, in the following theorems an application to the Sotomayor's theorem is appropriate**.**

Now, according to Jacobian matrix of system (2.2) given in [4], it is clear to verify that for any non - zero vector $V = (v_1,\ v_2,\ v_3,\ v_4)^T$ we have:

$$D^2F(X,\mu)(V,V) = \begin{bmatrix} -2v_1\left(v_1 - \frac{c_3c_2z}{R^3}v_1 + (1+c_1)v_2 + \frac{c_3c_2}{R^2}v_3\right) \\ 2v_2\,(c_1v_1 - c_4v_3 - c_5v_4) \\ \left(-\frac{2c_3c_8z}{R^3}v_1^2 + \frac{2c_3c_8}{R^2}v_1v_3 + 2c_9v_2v_3 - 2c_{10}v_3v_4\right) \\ 2v_4\,(c_{12}v_2 + c_{10}v_3) \end{bmatrix}, \tag{3.1}$$

and

$$D^3F(X,\mu)(V,V) = \begin{bmatrix} -\frac{6c_2c_3z}{R^4}v_1^3 + \frac{6c_2c_3}{R^3}v_3v_1^2 \\ 0 \\ \frac{6c_3c_8z}{R^4}v_1^3 - \frac{6c_3c_8}{R^3}v_3v_1^2 \\ 0 \end{bmatrix}, \tag{3.2}$$

where $R = (x + c_3)$ and $X = (x, y, z, w)$, μ be any bifurcation parameter.
In the following theorems the local bifurcation conditions near equilibrium points are established.

Theorem (3.1):
System (2.2) at the equilibrium point $E_1 = (1\,,0\,,0\,,0\,)$ with the parameter $c_{11} = \hat{c}_{11} = \frac{c_8}{\hat{R}}$ where $\hat{R} = (1 + c_3)$ has:

◊ No saddle –node bifurcation.
◊ Transcritical bifurcation .
◊ No pitch fork bifurcation.

Proof: According to the Jacobian matrix J_1 given in[4] the system (2.2) at the equilibrium point E_1 has zero eigenvalue (say $\lambda_{1z} = 0$) at $c_{11} = \hat{c}_{11}$, it is clear that $\hat{c}_{11} > 0$, and the Jacobian matrix J_1 with $c_{11} = \hat{c}_{11}$ becomes:

$$\hat{J}_1 = J_1(\lambda_{1z} = 0) = \begin{bmatrix} -1 & -(1+c_1) & -\frac{c_2}{\hat{R}} & 0 \\ 0 & c_1 - (c_6 + c_7) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -(\hat{c}_{11} + c_{13} + c_{14}) \end{bmatrix}$$

Now, let $V^{[1]} = \left(v_1^{[1]}, v_2^{[1]}, v_3^{[1]}, v_4^{[1]}\right)^T$ be the eigenvector corresponding to the eigenvalue $\lambda_{1z} = 0$. Thus $(\hat{J}_1 - \lambda_{1z}I)V^{[1]} = 0$, which gives:

$$v_1^{[1]} = \frac{-c_2}{\hat{R}}v_3^{[1]}, \qquad v_2^{[1]} = v_4^{[1]} = 0$$

and $v_3^{[1]}$ is any nonzero real number. Let $B^{[1]} = \left(b_1^{[1]}, b_2^{[1]}, b_3^{[1]}, b_4^{[1]}\right)^T$ be the eigenvector associated with the eigenvalue $\lambda_{1z} = 0$ of the matrix $\hat{J}_1^T$. Then we have, $\left(\hat{J}_1^T - \lambda_{1z}I\right)B^{[1]} = 0$. By solving this equation for $B^{[1]}$ we obtain, $B^{[1]} = \left(0\,,0\,,b_3^{[1]}\,,0\right)^T$, where $b_3^{[1]}$ is any nonzero real number. Now, consider:

$$\frac{\partial f}{\partial c_{11}} = f_{c_{11}}(X\,,c_{11}) = \left(\frac{\partial f_1}{\partial c_{11}},\frac{\partial f_2}{\partial c_{11}},\frac{\partial f_3}{\partial c_{11}},\frac{\partial f_4}{\partial c_{11}}\right)^T = (0\,,0\,,-z,-w)^T\ .$$

So, $f_{c_{11}}(E_1\,,\hat{c}_{11}) = (0\,,0\,,0\,,0)^T$ and hence $\left(B^{[1]}\right)^T f_{c_{11}}(E_1,\hat{c}_{11}) = 0$.
Therefore, by using Sotomayor's theorem the saddle-node bifurcation condition can not satisfy. While the first condition of transcritical bifurcation is satisfied, as below, since

$$Df_{c_{11}}(X\,,c_{11}) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

where $Df_{c_{11}}(X\,,c_{11})$ represents the derivative of $f_{c_{11}}(X\,,c_{11})$ with respect to $X = (x\,,y\,,z,w)^T$ .Further ,it is observed that

$$Df_{c_{11}}(E_1,\hat{c}_{11})V^{[1]}=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&-1&0\\0&0&0&-1\end{bmatrix}\begin{bmatrix}-\frac{c_2}{\hat{R}}v_3^{[1]}\\0\\v_3^{[1]}\\0\end{bmatrix}=\begin{bmatrix}0\\0\\-v_3^{[1]}\\0\end{bmatrix}$$

$$\left(B^{[1]}\right)^T\left[Df_{c_{11}}(E_1,\hat{c}_{11})V^{[1]}\right]=\left(0,0,b_3^{[1]},0\right)\left(0,0,-v_3^{[1]},0\right)^T=-v_3^{[1]}b_3^{[1]}\neq 0$$

Moreover, by substituting $V^{[1]}$ in (3.1) we get:

$$D^2f(E_1,\hat{c}_{11})\left(V^{[1]},V^{[1]}\right)=\begin{bmatrix}-2\left(\frac{c_2v_3^{[1]}}{\hat{R}}\right)^2+\frac{2c_3c_2^2}{\hat{R}^3}\left(v_3^{[1]}\right)^2\\0\\-\frac{2c_2c_3c_8}{\hat{R}^3}\left(v_3^{[1]}\right)^2\\0\end{bmatrix}$$

Hence, it is obtain that:

$$\left(B^{[1]}\right)^T\left[D^2f(E_1,\hat{c}_{11})\left(V^{[1]},V^{[1]}\right)\right]=-\frac{2c_2c_3c_8}{\hat{R}^3}b_3^{[1]}\left(v_3^{[1]}\right)^2\neq 0$$

Thus, by using Sotomayor's theorem system (2.2) has transcritical bifurcation at E_1 with the parameter $c_{11}=\hat{c}_{11}$, and no pitch fork bifurcation can occurs at $c_{11}=\hat{c}_{11}$ ■

Theorem (3.2): Suppose that the following condition

$$c_{12}\bar{y}>(c_{13}+c_{14}) \qquad (3.2a)$$

is satisfied. Then system (2.2) at the equilibrium point $E_2=(\bar{x},\bar{y},0,0)$ with the parameter $c_{11}=\bar{c}_{11}=c_{12}\bar{y}-(c_{13}+c_{14})$ has:

◊ No saddle –node bifurcation.
◊ Transcritical bifurcation.
◊ No pitch fork bifurcation

Proof: According to the Jacobian matrix J_2 given in[4] the system (2.2) at the equilibrium point E_2 has zero eigenvalue (say $\lambda_{2w}=0$) at $c_{11}=\bar{c}_{11}$, it is clear that $\bar{c}_{11}>0$ provided that the condition $(3.2a)$ holds, and the Jacobian matrix J_2 with $c_{11}=\bar{c}_{11}$ becomes:

$$\bar{J}_2=J_2(\bar{c}_{11})=\left[\bar{k}_{ij}\right]_{4\times4}$$

where $\bar{k}_{ij}=k_{ij}$ for all $i,j=1,2,3,4$ except $\bar{k}_{44}=0.$

Now, let $V^{[2]}=\left(v_1^{[2]},v_2^{[2]},v_3^{[2]},v_4^{[2]}\right)^T$ be the eigenvector corresponding to the eigenvalue $\lambda_{2w}=0$. Thus $(\bar{J}_2-\lambda_{2w}I)V^{[2]}=0$, which gives:

$$v_1^{[2]}=\frac{c_5}{c_1}v_4^{[2]}\ ,v_2^{[2]}=-\frac{c_5v_4^{[2]}}{c_1(c_1+1)},\ \ v_3^{[2]}=0,$$

and $v_4^{[2]}$ is any nonzero real number. Let $B^{[2]}=\left(b_1^{[2]},b_2^{[2]},b_3^{[2]},b_4^{[2]}\right)^T$ be the eigenvector associated with the eigenvalue $\lambda_{2w}=0$ of the matrix $\bar{J}_2^T$. Then we have, $(\bar{J}_2^T-\lambda_{2w}I)B^{[2]}=0$. By solving this equation for $B^{[2]}$ we obtain, $B^{[2]}=\left(0,0,0,b_4^{[2]}\right)^T$, where $b_4^{[2]}$ is any nonzero real number. Now, consider:

$$\frac{\partial f}{\partial c_{11}}=f_{c_{11}}(X,c_{11})=\left(\frac{\partial f_1}{\partial c_{11}},\frac{\partial f_2}{\partial c_{11}},\frac{\partial f_3}{\partial c_{11}},\frac{\partial f_4}{\partial c_{11}}\right)^T=(0,0,-z,-w)^T\ .$$

So, $f_{c_{11}}(E_2,\bar{c}_{11})=(0,0,0,0)^T$ and hence $\left(B^{[2]}\right)^Tf_{c_{11}}(E_2,\bar{c}_{11})=0$.

Therefore, by using Sotomayor's theorem the saddle-node bifurcation condition can not satisfy. While the first condition of transcritical bifurcation is satisfied. Now, since

$$Df_{c_{11}}(X, c_{11}) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

where $Df_{c_{11}}(X, \bar{c}_{11})$ represents the derivative of $f_{c_{11}}(X, \bar{c}_{11})$ with respect to $X = (x, y, z, w)^T$ .Further ,it is observed that

$$Df_{c_{11}}(E_2, \bar{c}_{11})V^{[2]} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{c_5}{c_1} v_4^{[2]} \\ -\frac{c_5 v_4^{[2]}}{c_1(c_1+1)} \\ 0 \\ v_4^{[2]} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -v_4^{[2]} \end{bmatrix}$$

$$\left(B^{[2]}\right)^T \left[Df_{c_{11}}(E_2, \bar{c}_{11})V^{[2]}\right] = \left(0, 0, 0, b_4^{[2]}\right)\left(0, 0, 0, -v_4^{[2]}\right)^T = -v_4^{[2]} b_4^{[2]} \neq 0$$

Moreover, by substituting $V^{[2]}$ in (3.1) we get:

$$D^2 f(E_2, \bar{c}_{11})\left(V^{[2]}, V^{[2]}\right) = \begin{bmatrix} -\frac{2c_5}{c_1}\left(v_4^{[2]}\right)^2 \left(\frac{c_5}{c_1} - (1+c_1).\frac{c_5}{c_1(c_1+1)}\right) \\ -2\frac{c_5}{c_1(c_1+1)}\left(v_4^{[2]}\right)^2 \left(\frac{c_1 c_5}{c_1} - c_5\right) \\ 0 \\ -2\frac{c_5 c_{12}}{c_1(1+c_1)}\left(v_4^{[2]}\right)^2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2\frac{c_5 c_{12}}{c_1(1+c_1)}\left(v_4^{[2]}\right)^2 \end{bmatrix}.$$

Hence, it is obtain that:

$$\left(B^{[2]}\right)^T \left[D^2 f(E_2, \bar{c}_{11})\left(V^{[2]}, V^{[2]}\right)\right] = -2\frac{c_5 c_{12}}{c_1(1+c_1)}\left(v_4^{[2]}\right)^2 b_4^{[2]} \neq 0$$

Thus, by using Sotomayor's theorem system (2.2) has transcritical bifurcation at E_2 with the parameter $c_{11} = \bar{c}_{11}$, and no pitch fork bifurcation can occurs at $c_{11} = \bar{c}_{11}$ ■

Theorem (3.3): Suppose that the following condition

$$c_{10}\dot{z} > (c_{13} + c_{14}) \qquad (3.3a)$$

is satisfied. Then system (2.2) at the equilibrium point $E_3 = (\dot{x}, 0, \dot{z}, 0)$ with the parameter $c_{11} = \dot{c}_{11} = c_{10}\dot{z} - (c_{13} + c_{14})$ has:

- ◊ No saddle –node bifurcation.
- ◊ Transcritical bifurcation.
- ◊ No pitch fork bifurcation.

Proof: According to the Jacobian matrix J_3 given in[4] the system (2.2) at the equilibrium point E_3 has zero eigenvalue (say $\lambda_{3w} = 0$) at $c_{11} = \dot{c}_{11}$, it is clear that $\dot{c}_{11} > 0$ provided that the condition $(3.3a)$ holds, and the Jacobian matrix J_2 with $c_{11} = \dot{c}_{11}$ becomes:

$$\dot{J}_3 = J_3(\dot{c}_{11}) = \left[\dot{z}_{ij}\right]_{4\times 4}$$

where $\dot{z}_{ij} = z_{ij}$ for all $i,j = 1,2,3,4$ except $\dot{z}_{44} = 0$.

Now, let $V^{[3]} = \left(v_1^{[3]}, v_2^{[3]}, v_3^{[3]}, v_4^{[3]}\right)^T$ be the eigenvector corresponding to the eigenvalue $\lambda_{3w} = 0$.
Thus $(\dot{J}_3 - \lambda_{3w} I)V^{[3]} = 0$, which gives:

$$v_1^{[3]} = -\frac{\dot{z}_{34}}{\dot{z}_{31}} v_4^{[3]}, \quad v_2^{[3]} = 0, \quad v_3^{[3]} = \frac{\dot{z}_{11}}{\dot{z}_{13}} \frac{\dot{z}_{34}}{\dot{z}_{31}} v_4^{[3]}$$

and $v_4^{[3]}$ is any nonzero real number. Let $B^{[3]} = \left(b_1^{[3]}, b_2^{[3]}, b_3^{[3]}, b_4^{[3]}\right)^T$ be the eigenvector associated with the eigenvalue $\lambda_{3w} = 0$ of the matrix $\dot{J}_3^T$. Then we have, $(\dot{J}_3^T - \lambda_{3w} I)B^{[3]} = 0$. By solving this equation for $B^{[3]}$ we obtain, $B^{[3]} = \left(0, 0, 0, b_4^{[3]}\right)^T$, where $b_4^{[3]}$ is any nonzero real number. Now, consider:

$$\frac{\partial f}{\partial c_{11}} = f_{c_{11}}(X, c_{11}) = \left(\frac{\partial f_1}{\partial c_{11}}, \frac{\partial f_2}{\partial c_{11}}, \frac{\partial f_3}{\partial c_{11}}, \frac{\partial f_4}{\partial c_{11}}\right)^T = (0, 0, -z, -w)^T .$$

So, $f_{c_{11}}(E_3, \dot{c}_{11}) = (0, 0, -\dot{z}, 0)^T$ and hence $\left(B^{[3]}\right)^T f_{c_{11}}(E_3, \dot{c}_{11}) = 0$.

Therefore, by using Sotomayor's theorem the saddle-node bifurcation condition can not satisfy. While the first condition of transcritical bifurcation is satisfied. Now, since

$$Df_{c_{11}}(X, c_{11}) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

where $Df_{c_{11}}(X, c_{11})$ represents the derivative of $f_{c_{11}}(X, c_{11})$ with respect to $X = (x, y, z, w)^T$ .Further ,it is observed that

$$Df_{c_{11}}(E_3, \dot{c}_{11})V^{[3]} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -\frac{\dot{z}_{34}}{\dot{z}_{31}} v_4^{[3]} \\ 0 \\ \frac{\dot{z}_{11}}{\dot{z}_{13}} \frac{\dot{z}_{34}}{\dot{z}_{31}} v_4^{[3]} \\ v_4^{[3]} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\frac{\dot{z}_{11}}{\dot{z}_{13}} \frac{\dot{z}_{34}}{\dot{z}_{31}} v_4^{[3]} \\ -v_4^{[3]} \end{bmatrix}$$

$$\left(B^{[3]}\right)^T \left[Df_{c_{11}}(E_3, \dot{c}_{11})V^{[3]}\right] = \left(0, 0, 0, b_4^{[3]}\right)\left(0, 0, -\frac{\dot{z}_{11}}{\dot{z}_{13}} \frac{\dot{z}_{34}}{\dot{z}_{31}} v_4^{[3]}, -v_4^{[3]}\right)^T = -v_4^{[3]} b_4^{[3]} \neq 0$$

Moreover, by substituting $V^{[3]}$ in (3.1) we get:

$$D^2 f(E_3, \dot{c}_{11})\left(V^{[3]}, V^{[3]}\right) = \begin{bmatrix} 2\left(\frac{\dot{z}_{34}}{\dot{z}_{31}} v_4^{[3]}\right)^2 \left(-1 + c_3 c_2 \frac{\dot{z}}{\dot{R}^3} + \frac{c_2 c_3}{\dot{R}^2} \frac{\dot{z}_{11}}{\dot{z}_{13}}\right) \\ 0 \\ 2\left(v_4^{[3]}\right)^2 \left(-\frac{c_3 c_8 \dot{z}}{\dot{R}^3}\left(\frac{\dot{z}_{34}}{\dot{z}_{31}}\right)^2 - \frac{c_8 c_3}{\dot{R}^2} \frac{\dot{z}_{11}}{\dot{z}_{13}}\left(\frac{\dot{z}_{34}}{\dot{z}_{31}}\right)^2 - c_{10} \frac{\dot{z}_{11}}{\dot{z}_{13}} \frac{\dot{z}_{34}}{\dot{z}_{31}}\right) \\ 2c_{10} \frac{\dot{z}_{11}}{\dot{z}_{13}} \frac{\dot{z}_{34}}{\dot{z}_{31}} \left(v_4^{[3]}\right)^2 \end{bmatrix}$$

Hence, it is obtain that:

$$\left(B^{[3]}\right)^T \left[D^2 f(E_3, \dot{c}_{11})\left(V^{[3]}, V^{[3]}\right)\right] = 2c_{10} \frac{\dot{z}_{11}}{\dot{z}_{13}} \frac{\dot{z}_{34}}{\dot{z}_{31}} \left(v_4^{[3]}\right)^2 b_4^{[3]} \neq 0$$

Thus, by using Sotomayor's theorem system (2.2) has transcritical bifurcation at E_3 with the parameter $c_{11} = \dot{c}_{11}$, and no pitch fork bifurcation can occurs at $c_{11} = \dot{c}_{11}$ ■

Theorem (3.4): Suppose that the following conditions (4.18)and (4.19)

$\Gamma_1 \neq \Gamma_2$ (3.4a)

$c_{10}\bar{\bar{z}} + c_{12}\bar{\bar{y}} > (c_{13} + c_{14})$, (3.4b) where ↓

$\Gamma_1 = c_{12}(d_{13}d_{21}d_{34} - d_{11}d_{23}d_{34}) + c_{10}\, d_{12}d_{31}d_{24}$ and

$\Gamma_2 = c_{12}d_{13}d_{24}d_{31} + c_{10}\,(d_{11}d_{32}d_{24} + d_{12}d_{21}d_{34})$

are satisfied. Then system (2.2) at the equilibrium point $E_4 = (\bar{\bar{x}}, \bar{\bar{y}}, \bar{\bar{z}}, 0)$ with the parameter $c_{11} = \bar{\bar{c}}_{11} = c_{10}\bar{\bar{z}} + c_{12}\bar{\bar{y}} - (c_{13} + c_{14})$ has:

◊ No saddle –node bifurcation.
◊ Transcritical bifurcation.
◊ No pitch fork bifurcation.

Proof: According to the Jacobian matrix J_4 given in[4]the system (2.2) at the equilibrium point E_4 has zero eigenvalue (say $\lambda_{4w} = 0$) at $c_{11} = \bar{\bar{c}}_{11}$, it is clear that $\bar{\bar{c}}_{11} > 0$ provided that the condition $(3.4b)$ holds, and the Jacobian matrix J_4 with $c_{11} = \bar{\bar{c}}_{11}$ becomes:

$\bar{\bar{J}}_4 = J_4(\bar{\bar{c}}_{11}) = [\bar{\bar{d}}_{ij}]_{4\times4}$,
where $\bar{\bar{d}}_{ij} = d_{ij}$ for all $i, j = 1,2,3,4$ except $\bar{\bar{d}}_{44} = 0$.
Now, let $V^{[4]} = \left(v_1^{[4]}, v_2^{[4]}, v_3^{[4]}, v_4^{[4]}\right)^T$ be the eigenvector corresponding to the eigenvalue $\lambda_{4w} = 0$. Thus $(\bar{\bar{J}}_4 - \lambda_{4w} I)V^{[4]} = 0$, which gives:

$$v_1^{[4]} = \frac{A}{U_3} v_4^{[4]}, \quad v_2^{[4]} = \frac{B}{U_3} v_4^{[4]}, \quad v_3^{[4]} = \frac{C}{U_3} v_4^{[4]},$$

where
$A = d_{13}d_{24}d_{32} + d_{12}d_{23}d_{34}$
$B = d_{13}d_{21}d_{34} - d_{11}d_{23}d_{34} - d_{13}d_{24}d_{31}$
$C = d_{24}(d_{12}d_{31} - d_{11}d_{32}) - d_{12}d_{21}d_{34}$
$U_3 = d_{11}d_{23}d_{32} - d_{12}d_{23}d_{31} - d_{13}d_{21}d_{32} > 0$, under the conditions of the stability (4.18) and (4.19), which are given in [4] ,and $v_4^{[4]}$ is any nonzero real number.

Let $B^{[4]} = \left(b_1^{[4]}, b_2^{[4]}, b_3^{[4]}, b_4^{[4]}\right)^T$ be the eigenvector associated with the eigenvalue $\lambda_{4w} = 0$ of the matrix $\bar{\bar{J}}_4^T$. Then we have, $(\bar{\bar{J}}_4^T - \lambda_{4w} I)B^{[4]} = 0$. By solving this equation for $B^{[4]}$ we obtain, $B^{[4]} = \left(0, 0, 0, b_4^{[4]}\right)^T$, where $b_4^{[4]}$ is any nonzero real number. Now, consider:

$$\frac{\partial f}{\partial c_{11}} = f_{c_{11}}(X, c_{11}) = \left(\frac{\partial f_1}{\partial c_{11}}, \frac{\partial f_2}{\partial c_{11}}, \frac{\partial f_3}{\partial c_{11}}, \frac{\partial f_4}{\partial c_{11}}\right)^T = (0, 0, -z, -w)^T .$$

So, $f_{c_{11}}(E_4, \bar{\bar{c}}_{11}) = (0, 0, -\bar{\bar{z}}, 0)^T$ and hence $\left(B^{[4]}\right)^T f_{c_{11}}(E_4, \bar{\bar{c}}_{11}) = 0$.

Therefore, by using Sotomayor's theorem the saddle-node bifurcation condition can not satisfy. While the first condition of transcritical bifurcation is satisfied. Now, since

$$Df_{c_{11}}(X, c_{11}) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

where $Df_{c_{11}}(X, c_{11})$ represents the derivative of $f_{c_{11}}(X, c_{11})$ with respect to $X = (x, y, z, w)^T$ .Further ,it is observed that

$$Df_{c_{11}}(E_4, \bar{\bar{c}}_{11})V^{[4]} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{A}{U_3} v_4^{[4]} \\ \frac{B}{U_3} v_4^{[4]} \\ \frac{C}{U_3} v_4^{[4]} \\ v_4^{[4]} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\frac{C}{U_3} v_4^{[4]} \\ -v_4^{[4]} \end{bmatrix}$$

$$\left(B^{[4]}\right)^T \left[Df_{c_{11}}(E_4, \bar{\bar{c}}_{11})V^{[4]}\right] = \left(0, 0, 0, b_4^{[4]}\right)\left(0, 0, -\frac{C}{U_3} v_4^{[4]}, -v_4^{[4]}\right)^T = -v_4^{[4]} b_4^{[4]} \neq 0$$

Moreover, by substituting $V^{[4]}$ in (3.1) we get:

$$D^2 f(E_4, \bar{\bar{c}}_{11})(\mathrm{V}^{[4]}, \mathrm{V}^{[4]}) = \begin{bmatrix} -2\frac{\mathrm{A}}{\mathrm{U}_3}\mathrm{v}_4^{[4]}\left(\frac{\mathrm{A}}{\mathrm{U}_3}\mathrm{v}_4^{[4]} - c_3c_2\bar{\bar{z}}\frac{\mathrm{A}}{\bar{\bar{\mathrm{R}}}^3\mathrm{U}_3}\mathrm{v}_4^{[4]} + (1+c_1)\frac{B}{U_3}\mathrm{v}_4^{[4]} + \frac{c_2c_3}{\bar{\bar{R}}^2}\frac{C}{U_3}\mathrm{v}_4^{[4]}\right) \\ 2\frac{B}{U_3}\left(\mathrm{v}_4^{[4]}\right)^2\left(\frac{c_1A}{U_3} - c_4\frac{C}{U_3} - c_5\right) \\ 2\left(\mathrm{v}_4^{[4]}\right)^2\left(-\frac{c_8c_3\bar{\bar{z}}}{\bar{\bar{R}}^3}\left(\frac{\mathrm{A}}{\mathrm{U}_3}\right)^2 + \frac{c_8c_3}{\bar{\bar{R}}^2}\left(\frac{AC}{U_3^2}\right) + \frac{c_9BC}{U_3^2} - \frac{c_{10}C}{U_3}\right) \\ 2\frac{\left(\mathrm{v}_4^{[4]}\right)^2}{U_3}\,(c_{12}B + c_{10}C) \end{bmatrix}$$

Hence, it is obtain that:

$$\left(B^{[4]}\right)^T\left[D^2 f(E_4, \bar{\bar{c}}_{11})(\mathrm{V}^{[4]}, \mathrm{V}^{[4]})\right] = 2\frac{\left(\mathrm{v}_4^{[4]}\right)^2}{U_3}\,(c_{12}B + c_{10}C)b_4^{[4]}$$

$$= 2\frac{\left(\mathrm{v}_4^{[4]}\right)^2}{U_3}\mathrm{b}_4^{[4]}(c_{12}(d_{13}d_{21}d_{34} - d_{11}d_{23}d_{34}) + c_{10}\,d_{12}d_{31}d_{24} - (c_{12}d_{13}d_{24}d_{31} + c_{10}\,(d_{11}d_{32}d_{24} + d_{12}d_{21}d_{34})\,)\,)$$

$$= 2\frac{\left(\mathrm{v}_4^{[4]}\right)^2}{U_3}\mathrm{b}_4^{[4]}(\Gamma_1 - \Gamma_2)$$

So, according to condition $(3.4a)$ and in addition to the conditions of the stability (4.18) and (4.19), which are given in [4] ,we obtain that:

$$\left(B^{[4]}\right)^T\left[D^2 f(E_4, \bar{\bar{c}}_{11})(\mathrm{V}^{[4]}, \mathrm{V}^{[4]})\right] \neq 0$$

Thus, by using Sotomayor's theorem system (2.2) has transcritical bifurcation at E_4 with the parameter $c_{11} = \bar{\bar{c}}_{11}$, on the other hand if the condition (3.4a) is relegation, then we get:

$$D^3 f(E_4, \bar{\bar{c}}_{11})(\mathrm{V}^{[4]}, \mathrm{V}^{[4]}, V^{[4]}) = \begin{bmatrix} \frac{\left(-\frac{6c_2c_3\bar{\bar{z}}A^2}{\bar{\bar{R}}^4U_3} + \frac{6c_2c_3}{\bar{\bar{R}}^3}C\right)A}{U_3^2}\left(v_4^{[4]}\right)^3 \\ 0 \\ \frac{6c_3c_8\bar{\bar{z}}}{\bar{\bar{R}}^4}\left(\frac{Av_4^{[4]}}{U_3}\right)^3 - \frac{\frac{6c_3c_8}{\bar{\bar{R}}^3}AC}{U_3^2}\left(v_4^{[4]}\right)^3 \\ 0 \end{bmatrix}$$

$$\left(B^{[4]}\right)^T\left[D^3 f(E_4, \bar{\bar{c}}_{11})(\mathrm{V}^{[4]}, \mathrm{V}^{[4]}, \mathrm{V}^{[4]})\right] = 0$$

So, there is no pitch fork bifurcation. ■

Theorem (3.5): Suppose that the following condition (4.26)

$c_1\tilde{x} > c_4\tilde{z} + c_5\widetilde{w} + c_7$ $\quad(3.5a)$

$c_{10}\tilde{z} < (c_{11} + c_{13} + c_{14})$ $\quad(3.5b)$

$\xi_1 \neq \xi_2$ $\quad(3.5c)$

where

$\xi_1 = c_1r_{13}r_{34}r_{42} + c_4(r_{11}r_{34}r_{42} + r_{12}r_{31}r_{44}) - c_5r_{12}r_{31}r_{43}$

and

$\xi_2 = c_1(\,r_{13}r_{32}r_{44} + r_{12}r_{34}r_{43}) + c_4r_{11}r_{32}r_{44} + c_5(r_{11}r_{32}r_{43} + r_{13}r_{31}r_{42})$

are satisfied. Then system (2.2) at the equilibrium point $E_5 = (\tilde{x}, 0, \tilde{z}, \widetilde{w})$ with the parameter $c_6 = \tilde{c}_6 = c_1\tilde{x} - (c_4\tilde{z} + c_5\widetilde{w} + c_7)$ has:

- ◊ No saddle –node bifurcation.
- ◊ Transcritical bifurcation.
- ◊ No pitch fork bifurcation.

Proof: According to the Jacobian matrix J_5 given in[4], the system (2.2) at the equilibrium point E_5 has zero eigenvalue (say $\lambda_{5y} = 0$) at $c_6 = \tilde{c}_6$, it is clear that $\tilde{c}_6 > 0$ provided that the condition $(3.5a)$ holds, and the Jacobian matrix $\tilde{J}_5$ with $c_6 = \tilde{c}_6$ becomes:

$\tilde{J}_5 = J_5(\tilde{c}_6) = \left[\tilde{r}_{ij}\right]_{4\times4}$,

where $\tilde{r}_{ij} = r_{ij}$ for all $i,j = 1,2,3,4$ except $\tilde{r}_{22} = 0$.

Now, let $\mathrm{V}^{[5]}=\left(\mathrm{v}_1^{[5]},\mathrm{v}_2^{[5]},\mathrm{v}_3^{[5]},\mathrm{v}_4^{[5]}\right)^T$ be the eigenvector corresponding to the eigenvalue $\lambda_{5\mathrm{y}}=0$. Thus $\left(\tilde{J}_5-\lambda_{5\mathrm{y}}I\right)\mathrm{V}^{[5]}=0$, which gives:

$\mathrm{v}_1^{[5]}=H_1\mathrm{v}_2^{[5]}$, $\mathrm{v}_3^{[5]}=H_2\mathrm{v}_2^{[5]}$, $\mathrm{v}_4^{[5]}=H_3\mathrm{v}_2^{[5]}$,
where

$$H_1=\frac{r_{13}r_{34}r_{42}-r_{13}r_{32}r_{44}-r_{12}r_{34}r_{43}}{\zeta},$$

$$H_2=\frac{-(r_{11}r_{34}r_{42}+r_{12}r_{31}r_{44})+r_{11}r_{32}r_{44}}{\zeta},$$

$$H_3=\frac{r_{12}r_{31}r_{43}-(r_{11}r_{32}r_{43}+r_{13}r_{31}r_{42})}{\zeta},$$

$\zeta=r_{11}r_{34}r_{43}+r_{31}r_{13}r_{44}>0$ under the condition of the stability (4.26) ,which is given in [4] and in addition $(3.5b)$,and $\mathrm{v}_2^{[5]}$ is any nonzero real number. Let $B^{[5]}=\left(b_1^{[5]},b_2^{[5]},b_3^{[5]},b_4^{[5]}\right)^T$ be the eigenvector associated with the eigenvalue $\lambda_{5\mathrm{y}}=0$ of the matrix $\tilde{J}_5^T$. Then we have, $\left(\tilde{J}_5^T-\lambda_{5\mathrm{y}}I\right)B^{[5]}=0$. By solving this equation for $B^{[5]}$ we obtain, $B^{[5]}=\left(0,\mathrm{b}_2^{[5]},0,0\right)^T$, where $b_2^{[5]}$ is any nonzero real number. Now, consider:

$$\frac{\partial f}{\partial c_6}=f_{c_6}(X,c_6)=\left(\frac{\partial f_1}{\partial c_6},\frac{\partial f_2}{\partial c_6},\frac{\partial f_3}{\partial c_6},\frac{\partial f_4}{\partial c_6}\right)^T=(0,-y,0,0)^T\ .$$

So, $f_{c_6}(E_5,\tilde{c}_6)=(0,0,0,0)^T$ and hence $\left(B^{[5]}\right)^T f_{c_6}(E_5,\tilde{c}_6)=0$.

Therefore, by using Sotomayor's theorem the saddle-node bifurcation condition can not satisfy. While the first condition of transcritical bifurcation is satisfied. Now, since

$$Df_{\mathrm{c}_6}(X,\tilde{c}_6)=\begin{bmatrix}0&0&0&0\\0&-1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

where $Df_{\mathrm{c}_6}(X,c_6)$ represents the derivative of $f_{\mathrm{c}_6}(X,c_6)$ with respect to $X=(x,\mathrm{y},\mathrm{z},w)^T$ .Further ,it is observed that

$$Df_{\mathrm{c}_6}(E_5,\tilde{c}_6)V^{[5]}=\begin{bmatrix}0&0&0&0\\0&-1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}H_1\mathrm{v}_2^{[5]}\\\mathrm{v}_2^{[5]}\\H_2\mathrm{v}_2^{[5]}\\H_3\mathrm{v}_2^{[5]}\end{bmatrix}=\begin{bmatrix}0\\-\mathrm{v}_2^{[5]}\\0\\0\end{bmatrix}$$

$$\left(B^{[5]}\right)^T\left[Df_{\mathrm{c}_6}\left(E_5,\mathrm{v}_2^{[5]}\right)\mathrm{V}^{[5]}\right]=\left(0,b_2^{[5]},0,0\right)\left(0,-\mathrm{v}_2^{[5]},0,0\right)^T=-b_2^{[5]}\mathrm{v}_2^{[5]}\neq 0$$

Moreover, by substituting $\mathrm{V}^{[5]}$ in (3.1) we get:

$$D^2f(E_5,\tilde{c}_6)\left(\mathrm{V}^{[5]},\mathrm{V}^{[5]}\right)=\begin{bmatrix}-2H_1\left(\mathrm{v}_2^{[5]}\right)^2\left(H_1\left(1-c_3c_2\frac{\widetilde{z}}{\tilde{R}^3}\right)+(c_1+1)+\frac{c_2c_3}{\tilde{R}^2}H_3\right)\\2\left(\mathrm{v}_2^{[5]}\right)^2(c_1H_1-c_4H_2-c_5H_3)\\-\frac{2c_3c_8\widetilde{z}}{\tilde{R}^3}\left(H_1\mathrm{v}_2^{[5]}\right)^2+\frac{2c_3c_8}{\widetilde{R^2}}H_1H_2\left(\mathrm{v}_2^{[5]}\right)^2+2H_2\left(\mathrm{v}_2^{[5]}\right)^2(c_9-c_{10}H_3)\\2H_3\left(\mathrm{v}_2^{[5]}\right)^2(c_{12}+c_{10}H_2)\end{bmatrix}$$

Hence, it is obtain that:

$$\left(B^{[5]}\right)^T\left[D^2f(E_5\,,\tilde{c}_6)\left(\mathrm{V}^{[5]}\,,\mathrm{V}^{[5]}\right)\right] = 2\left(\mathrm{v}_2^{[5]}\right)^2(c_1H_1 - c_4H_2 - c_5H_3)b_2^{[5]}$$

$$= 2\left(\mathrm{v}_2^{[5]}\right)b_2^{[5]}[c_1r_{13}r_{34}r_{42} + c_4(r_{11}r_{34}r_{42} + r_{12}r_{31}r_{44}) - c_5r_{12}r_{31}r_{43}$$

$$-\left(c_1(\,r_{13}r_{32}r_{44} + r_{12}r_{34}r_{43}) + c_4r_{11}r_{32}r_{44} - c_5(r_{11}r_{32}r_{43} + r_{13}r_{31}r_{42})\right)] = 2\left(\mathrm{v}_2^{[5]}\right)b_2^{[5]}(\xi_1 - \xi_2)$$

So, according to conditions(3.5a), (3.5b), (3.5c) and in addition to the condition of the stability (4.26) given in [4], we obtain that:

$$\left(B^{[5]}\right)^T\left[D^2f(E_5\,,\tilde{c}_6)\left(\mathrm{V}^{[5]}\,,\mathrm{V}^{[5]}\right)\right] \neq 0$$

Thus, by using Sotomayor's theorem system (2.2) has transcritical bifurcation at E_5 with the parameter $\mathrm{c}_6 = \tilde{\mathrm{c}}_6$, on the other hand if the condition (3.5c) is relegation ,then we get:

$$\left(B^{[5]}\right)^T\left[D^3f(E_5\,,\tilde{c}_6)\left(\mathrm{V}^{[5]}\,,\mathrm{V}^{[5]},\mathrm{V}^{[5]}\right)\right] = \begin{bmatrix} -\dfrac{6c_2c_3\tilde{z}}{\tilde{R}^4}\left(H_1v_2^{[5]}\right)^3 + \dfrac{6c_2c_3}{\tilde{R}^3}H_1^2H_2\left(v_2^{[5]}\right)^2 \\ 0 \\ \dfrac{6c_3c_8H_1^2}{\tilde{R}^3}\left(v_2^{[5]}\right)^2\left(\dfrac{H_1\tilde{z}}{\tilde{R}} - H_2\right) \\ 0 \end{bmatrix}$$

$$\left(B^{[5]}\right)^T\left[D^3f(E_5\,,\tilde{c}_6)\left(\mathrm{V}^{[5]}\,,\mathrm{V}^{[5]},\mathrm{V}^{[5]}\right)\right] = 0$$

So, there is no pitch fork bifurcation ■

Theorem (3.6): Suppose that the following conditions

$$c_{10}(1 + c_1) < \frac{c_2c_{12}}{c_3 + x^*} \qquad (3.6a)$$

$$\frac{c_8l_{24}}{l_{34}} > -\frac{c_2(l_{31}l_{24} - l_{21}l_{34})}{l_{11}l_{34}} \qquad (3.6b)$$

$$\Lambda_1 \neq \Lambda_2\,, \qquad (3.6c)$$

where

$$\Lambda_1 = -P_1^2\frac{(l_{31}l_{24} - l_{21}l_{34})}{l_{11}l_{34}} + \left(\frac{c_2(l_{31}l_{24} - l_{21}l_{34})}{l_{11}l_{34}} + \frac{c_8l_{24}}{l_{34}}\right)\left(\frac{c_3z^*P_1^2}{R^{*3}l_{43}} + \frac{c_3l_{42}P_1}{R^{*2}l_{43}}\right) + P_1\left(-(1 + c_1)\frac{(l_{31}l_{24} - l_{21}l_{34})}{l_{11}l_{34}} + c_1\right)$$
$$+ \frac{l_{42}}{l_{43}}\left(c_4 + \frac{l_{24}}{l_{34}}c_9\right) - \frac{l_{23}}{l_{43}}c_{12}P_2 + \frac{c_{10}}{l_{43}^2}P_2l_{13}\left(\frac{(l_{31}l_{24} - l_{21}l_{34})}{l_{11}l_{34}}\right),$$

and

$$\Lambda_2 = P_2\left(c_5 + \frac{c_{12}l_{13}}{l_{43}}\left(\frac{(l_{31}l_{24} - l_{21}l_{34})}{l_{11}l_{34}}\right)\right) - \frac{c_{10}P_2l_{42}}{l_{43}}\left(\frac{l_{23}}{l_{43}} - \frac{l_{24}}{l_{34}}\right),$$

$$P_1 = \frac{l_{13}l_{42} - l_{43}l_{12}}{l_{11}l_{43}} \quad and \quad P_2 = -\frac{1}{l_{34}}(l_{31}P_1 + l_{32}),$$

are satisfied. Then for the parameter value :

$$c_4^* = \frac{1}{c_{10}c_{12}\left(1 - \frac{c_2z^*}{R^{*2}}\right)}\left[c_{10}^2c_1(c_1 + 1) + \frac{c_2c_{12}}{R^{*3}}\left(c_5c_3c_8 - c_1c_{10}R^{*2}\right) + c_5c_{10}\left(c_9\left(1 - \frac{c_2z^*}{R^{*2}}\right) - \frac{c_3c_8(c_1 + 1)}{R^{*2}}\right)\right]$$

The system (2.2) at the equilibrium point $E_6 = (x^*, y^*, z^*, w^*)$ has saddle –node bifurcation, but neither transcritical bifurcation, nor pitch fork bifurcation can occur at E_6

proof: The characteristic equation of Jacobian matrix J_6 given in[4] having zero eigenvalue (say $\lambda_{6y} = 0$) if and only if $N_4 = 0$ and ,then E_6 becomes a non-hyperbolic equilibrium point. Clearly the Jacobian matrix of system (2.2) at the equilibrium point E_6 with parameter $c_4 = c_4^*$ becomes:

$J_6^* = J_6(c_4^*) = [\,l_{ij}^*\,]_{4\times4}$,

where $l^*_{ij} = l_{ij}$ for all $i,j = 1,2,3,4$ except $l_{23}^* = -c_4^*y^*$. Note that , $c_4^* > 0$ under the conditions of the stability (4.29),(4.32) and (4.33), which are given in [4]. Now, let $\mathrm{V}^{[6]} = \left(\mathrm{v}_1^{[6]}\,,\mathrm{v}_2^{[6]}\,,\mathrm{v}_3^{[6]}\,,\mathrm{v}_4^{[6]}\right)^T$ be the eigenvector corresponding to the eigenvalue $\lambda_{6\mathrm{y}} = 0$. Thus $\left(J^*{}_6 - \lambda_{2\mathrm{y}}I\right)\mathrm{V}^{[6]} = 0$, which gives:

$$\mathrm{v}_1^{[6]} = P_1\mathrm{v}_2^{[6]}, \quad \mathrm{v}_3^{[6]} = -\frac{l_{42}}{l_{43}}\mathrm{v}_2^{[6]}, \quad \mathrm{v}_4^{[6]} = P_2\mathrm{v}_2^{[6]}$$

and $v_2^{[5]}$ is any nonzero real number. Let $B^{[6]} = \left(b_1^{[6]}, b_2^{[6]}, b_3^{[6]}, b_4^{[6]}\right)^T$ be the eigenvector associated with the eigenvalue $\lambda_{6y} = 0$ of the matrix J_6^{*T}. Then we have, $\left(J_6^{*T} - \lambda_{6y} I\right)B^{[6]} = 0$. By solving this equation for $B^{[6]}$ we obtain,

$$b_1^{[6]} = \frac{l_{31}l_{24} - l_{21}l_{34}}{l_{11}l_{34}} b_2^{[6]}, b_3^{[6]} = -\frac{l_{24}}{l_{34}} b_2^{[6]},$$
$$b_4^{[6]} = -\frac{1}{l_{43}}\left[l_{13}\left(\frac{l_{31}l_{24} - l_{21}l_{34}}{l_{11}l_{34}}\right) + l_{23}\right) b_2^{[6]}$$

and $b_2^{[6]}$ is any nonzero real number. Now, consider:

$$\frac{\partial f}{\partial c_4} = f_{c_4}(X, c_4) = \left(\frac{\partial f_1}{\partial c_4}, \frac{\partial f_2}{\partial c_4}, \frac{\partial f_3}{\partial c_4}, \frac{\partial f_4}{\partial c_4}\right)^T = (0, -y\,z, 0, 0)^T.$$

$f_{c_4}(E_6, c_4^*) = (0, -z^* y^*, 0, 0)^T$ and hence $\left(B^{[6]}\right)^T f_{c_4}(E_6, c_4^*) = -z^* y^* b_2^{[6]} \neq 0$. Therefore, by using Sotomayor's theorem the transcritical and pitchfork bifurcation cannot occur. While the first condition of saddle-node bifurcation is satisfied. Now, by substituting $V^{[6]}$ in Eq (3.1) we get:

$$D^2 f(E_6, c_4^*)\left(V_6^{[6]}, V_6^{[6]}\right) = \left(h_{ij}\right)_{4\times 1}$$
$$h_{11} = -2P_1\left(v_2^{[6]}\right)^2\left(P_1\left(1 - \frac{c_3 c_2 z^*}{R^{*3}}\right) + (1 + c_1) - \frac{c_2 c_3 l_{42}}{R^{*2} l_{43}}\right)$$
$$h_{21} = 2\left(v_2^{[6]}\right)^2\left(c_1 P_1 + \frac{c_4 l_{42}}{l_{43}} - c_5 P_2\right)$$
$$\square_{31} = \left(-\left(\frac{2c_3 c_8 z^* P_1^2}{R^{*3}} + \frac{2c_3 c_8 l_{42} P_1}{R^{*2} l_{43}} + \frac{2c_9 l_{42}}{l_{43}}\right) + \frac{2c_{10} P_2 l_{42}}{l_{43}}\right)\left(v_2^{[6]}\right)^2$$
$$\square_{41} = 2P_2\left(v_2^{[6]}\right)^2\left(c_{12} - \frac{c_{10} l_{42}}{l_{43}}\right)$$

Now,

$$\left(B_6^{[6]}\right)^T\left(D^2 f(E_6, c_4^*)\left(V_6^{[6]}, V_6^{[6]}\right)\right)$$
$$= 2b_2^{[6]}\left(v_2^{[6]}\right)^2\left[-P_1^2\frac{(l_{31}l_{24} - l_{21}l_{34})}{l_{11}l_{34}} + \left(\frac{c_2(l_{31}l_{24} - l_{21}l_{34})}{l_{11}l_{34}} + \frac{c_8 l_{24}}{l_{34}}\right)\left(\frac{c_3 z^* P_1^2}{R^{*3} l_{43}} + \frac{c_3 l_{42} P_1}{R^{*2} l_{43}}\right)\right.$$
$$+ P_1\left(-(1 + c_1)\frac{(l_{31}l_{24} - l_{21}l_{34})}{l_{11}l_{34}} + c_1\right) + \frac{l_{42}}{l_{43}}\left(c_4^* + \frac{l_{24}}{l_{34}} c_9\right) - \frac{l_{23}}{l_{43}} c_{12} P_2 + \frac{c_{10}}{l_{43}^2} P_2 l_{13}\left(\frac{(l_{31}l_{24} - l_{21}l_{34})}{l_{11}l_{34}}\right)$$
$$\left. - P_2\left(c_5 + \frac{c_{12} l_{13}}{l_{43}}\left(\frac{(l_{31}l_{24} - l_{21}l_{34})}{l_{11}l_{34}}\right)\right) + \frac{c_{10} P_2 l_{42}}{l_{43}}\left(\frac{l_{23}}{l_{43}} - \frac{l_{24}}{l_{34}}\right)\right] = 2b_2^{[6]}\left(v_2^{[6]}\right)^2(\Lambda_1 - \Lambda_2) \neq 0$$

provided that the conditions $(3.6a) - (3.6c)$ in addition to the conditions of the stability (4.29),(4.32) and (4.33), which are given in [4]. Therefore, according to Sotomayors theorem the saddle node bifurcation occur at E_6 ■

4 Hopf bifurcation analysis:

To discuss the occurrence of Hopf bifurcation , first we need to know that the Hopf bifurcation for $n = 4$ are constructed according to the Haque and Venturino methods [10]. Consider the characteristic equation given by:

$$P_4(\tau) = \tau^4 + C_1\tau^3 + C_2\tau^2 + C_3\tau + C_4 = 0$$

here $C_1 = -tr\left(J\left(x^*\right)\right)$, $C_2 = M_1\left(J\left(x^*\right)\right)$, $C_3 = -M_2\left(J\left(x^*\right)\right)$ and $C_4 = \det\left(J\left(x^*\right)\right)$ with $M_1\left(J\left(x^*\right)\right)$ and $M_2\left(J\left(x^*\right)\right)$ represent the sum of the principal minors of order two and three of $J\left(x^*\right)$ respectively.

Clearly, the first condition of Hopf bifurcation holds if and only if:

$$C_i > 0;\ i = 1,3;\ \Delta_1 = C_1 C_2 - C_3 > 0;\ C_1^3 - 4\Delta_1 > 0$$
$$\Delta_2 = C_3\left(C_1 C_2 - C_3\right) - C_1^2 C_4 = 0$$

Consequently, . $C_4 = \frac{C_3(C_1C_2 - C_3)}{C_1^2}$ So, the characteristic equation becomes:

$$P_4(\tau) = \left(\tau^2 + \frac{C_3}{C_1}\right)\left(\tau^2 + C_1\tau + \frac{\Delta_1}{C_1}\right) = 0 \tag{1.31}$$

Clearly, the roots of eq. (1.31) are :

$$\tau_{1,2} = \frac{1}{2}\left(-C_1 \pm \sqrt{C_1^2 - 4\frac{\Delta_1}{C_1}}\right),\ \tau_{3,4} = \pm i\sqrt{\frac{C_3}{C_1}}$$

Now, to verify the transversality condition of Hopf bifurcation, we substitute $\tau(\eta) = \varsigma_1(\eta) \mp i\varsigma_2(\eta)$ into eq. (1.31), and then calculating its derivative with respect to the bifurcation parameter η , $P_4'(\tau(\eta)) = 0$ comparing the two sides of this equation and then equating their real and imaginary parts, we have:

$$\left.\begin{aligned} &\bar{\Psi}(\eta)\, \varsigma_1'(\eta) - \bar{\Phi}(\eta)\, \varsigma_2'(\eta) + \bar{\Theta}(\eta) = 0 \\ &\bar{\Phi}(\eta)\, \varsigma_1'(\eta) + \bar{\Psi}(\eta)\, \varsigma_2'(\eta) + \bar{\Gamma}(\eta) = 0 \end{aligned}\right\} \tag{1.32}$$

where

$$\left.\begin{aligned} &\bar{\Psi}(\eta) = 4\,(\varsigma_1(\eta))^3 + 3C_1(\eta)(\varsigma_1(\eta))^2 + C_3(\eta) + 2C_2(\eta)\varsigma_1(\eta) \\ &-12\,\varsigma_1(\eta)\,\varsigma_2^2(\eta) - 3C_1(\eta)(\varsigma_2(\eta))^2 \\ &\bar{\Phi}(\eta) = 12\,(\varsigma_1(\eta))^2\varsigma_2(\eta) + 6C_1(\eta)\,\varsigma_1(\eta)\,\varsigma_2(\eta) + 2\,C_2(\eta)\varsigma_2(\eta) \\ &-4\,(\varsigma_2(\eta))^3 \\ &\bar{\Theta}(\eta) = (\varsigma_1(\eta))^3C_1'(\eta) + C_3'(\eta)\,\varsigma_1(\eta) + C_2'(\eta)(\varsigma_1(\eta))^2 + C_4'(\eta) \\ &-3\,C_1'(\eta)\,\varsigma_1(\eta)(\varsigma_2(\eta))^2 - C_2'(\eta)(\varsigma_2(\eta))^2 \\ &\bar{\Gamma}(\eta) = 3\,(\varsigma_1(\eta))^2\varsigma_2(\eta)C_1'(\eta) + C_3'(\eta)\varsigma_2(\eta) + 2C_2'(\eta)\,\varsigma_1(\eta)\,\varsigma_2(\eta) \\ &-C_1'(\eta)(\varsigma_2(\eta))^3 \end{aligned}\right\} \quad 1.33)$$

Solving the linear system (1.32) by using Cramer's rule for the unknowns $\varsigma_1'(\eta)$ and $\varsigma_2'(\eta)$, gives that:

$$\varsigma_1'(\eta) = -\frac{\bar{\Theta}(\eta)\bar{\Psi}(\eta) + \bar{\Gamma}(\eta)\bar{\Phi}(\eta)}{(\bar{\Psi}(\eta))^2 + (\bar{\Phi}(\eta))^2} \text{ and } \varsigma_2'(\eta) = \frac{-\bar{\Gamma}(\eta)\bar{\Psi}(\eta) + \bar{\Theta}(\eta)\bar{\Phi}(\eta)}{(\bar{\Psi}(\eta))^2 + (\bar{\Phi}(\eta))^2}$$

Hence the transversality condition not being zero if and only if:

$$\bar{\Theta}(\eta)\bar{\Psi}(\eta) + \bar{\Gamma}(\eta)\bar{\Phi}(\eta) \neq 0 \tag{1.34}$$

Theorem (4.1): Suppose that the following conditions $(4.29) - (4.34)$ and in addition to the following condition:

$$N_3 < \Delta_1 < \frac{N_1^3}{4} \tag{4.1a}$$

$$c_{10}c_9z^* > c_{12}N_1 \tag{4.1b}$$

Where $\Delta_1 = -l_{11}(p_4 + p_5) - (p_6 + p_7 + p_{10})$, are satisfied, then at the parameter $c_5 = c_5^\star$,the system (2.2) has a Hopf bifurcation near the point E_6

Proof: Consider the characteristic equation of system (2.2) at E_6 which is given in[4], Then by using the Hopf bifurcation theorem for $n = 4$, we need to find a parameter say $(c_5^\star)$ to verify the necessary and sufficient conditions for Hopf bifurcation to occur satisfy that: $N_i(c_5^\star) > 0;\ i = 1;3$, $\Delta_1(c_5^\star) > 0$, $N_1^3(c_5^\star) - 4\Delta_1(c_5^\star) > 0$ and $\Delta_2(c_5^\star) = 0$.
Where $N_i;\ i = 1,3$ represent the coefficients of characteristic given in[4] straight forward computation gives that: $N_i(c_5^\star) > 0$; $i = 1,3$, $N_1 > 0$ provided that the condition (4.29) which given in [4] and $N_3 > 0$, $\Delta_1(c_5^\star) > 0$ provided that conditions of the stability (4.29) – (4.32) and (4.34) are hold, which are given in [4], While $N_1^3(c_5^\star) - 4\Delta_1(c_5^\star) > 0$ provided that condition $(4.1b)$ holds. On the other hand, it is observed that $\Delta_2 = 0$ gives:

$$\varphi_1c_5^2 + \varphi_2c_5 + \varphi_3 = 0 \tag{4.1c}$$

where

$\varphi_1 = l_{32}l_{43}y^*(l_{11}l_{42} - l_{32}l_{43}) < 0$

$\varphi_2 = [l_{11}^2 l_{42}p_4 - l_{32}l_{43}p_6 + l_{11}l_{42}p_6 + l_{11}^2(l_{11}l_{32}l_{43} - l_{12}l_{31}l_{43}) - p_7(l_{32}l_{43} - l_{11}l_{42}) + l_{32}l_{43}(N_1(p_4+p_5) - [(p_6 - l_{11}p_1) + p_7 - l_{11}p_2])\,]y^*$

$\varphi_3 = l_{11}^2\big(p_2p_4 + p_5(p_1+p_2)\big) - p_6\big((p_6 - l_{11}p_1) + p_7 - l_{11}p_2\big) + l_{11}^2 p_9 + l_{11}p_6\big(l_{11}^2 - (p_4+p_5)\big) + p_7(N_1(p_4+p_5) - (p_6 - l_{11}p_1) + p_7 - l_{11}p_2)$

it is easy to verify that, the eq. $(4.1c)$ has a unique positive root

$$c_5^\star = \frac{-1}{2\varphi_1}\left(\varphi_2 + \sqrt{\varphi_2^2 - 4\,\varphi_1\varphi_3}\right)$$

provided that conditions (4.29) – (4.32) and (4.34) given in[4] , Now, at $c_5 = c_5^\star$ the characteristic equation given in.[4] can be written as:

$$\left(\lambda_6^2 + \frac{N_3}{N_1}\right)\left(\lambda_6^2 + N_1\lambda_6 + \frac{\Delta_1}{N_1}\right) = 0 \quad ,$$

which has four roots $\lambda_{6\,x,y} = \pm i\sqrt{\frac{N_3}{N_1}}$ and $\lambda_{6\,z,w} = \frac{1}{2}\left(-N_1 \pm \sqrt{N_1^2 - 4\frac{\Delta_1}{N_1}}\right)$.

Clearly, at $c_5 = c_5^\star$ there are two pure imaginary eigenvalues ($\lambda_{6\,x}$ and λ_{6y}) and two eigenvalues which are real and negative $(4.1a)$. Now for all values of c_5 in the neighborhood of $c_5^\star$, the roots in general of the following form:

$$\lambda_{6x} = \delta_1 + i\delta_2 \ , \lambda_{6y} = \delta_1 - i\delta_2 \ , \lambda_{6\,z,w} = \frac{1}{2}\left(-N_1 \pm \sqrt{N_1^2 - 4\frac{\Delta_1}{N_1}}\right).$$

Clearly, $Re\left(\lambda_{6\,x,y}(c_5)\right)\big|_{c_5 = c_5^\star} = \delta_1(c_5^\star) = 0$ that means the first condition of the necessary and sufficient conditions for Hopf bifurcation is satisfied at $c_5 = c_5^\star$ .Now, according to verify the transversality condition we must prove that:

$$\overline{\Theta}(c_5^\star)\,\overline{\Psi}(c_5^\star) + \overline{\Gamma}(c_5^\star)\,\overline{\Phi}(c_5^\star) \neq 0 \quad ,$$

where $\overline{\Theta}, \overline{\Psi}$, $\overline{\Gamma}$ and $\overline{\Phi}$ are given in (1.33). Note that for $c_5 = c_5^\star$ we have $\delta_1 = 0$ and $\delta_2 = \sqrt{\frac{N_3}{N_1}}$, substituting the value of (δ_2) gives the following simplifications:

$$\overline{\Psi}(c_5^\star) = -2\,N_3(c_5^\star) \ ,$$

$$\overline{\Phi}(c_5^\star) = 2\,\frac{\delta_2(c_5^\star)}{N_1}(N_1N_2 - 2\,N_3) \ ,$$

$$\overline{\Theta}(c_5^\star) = N_4'(c_5^\star) - \frac{N_3}{N_1}N_2'(c_5^\star) \ ,$$

$$\overline{\Gamma}(c_5^\star) = \delta_2(c_5^\star)\left(N_3'(c_5^\star) - \frac{N_3}{N_1}N_1'(c_5^\star)\right) \ ,$$

where

$$N_1' = \frac{dN_1}{dc_5}\Big|_{c_5 = c_5^\star} = 0 \ ,$$

$$N_2' = \frac{dN_2}{dc_5}\Big|_{c_5 = c_5^\star} = l_{42}y^* \ ,$$

$$N_3' = \frac{dN_3}{dc_5}\Big|_{c_5 = c_5^\star} = (l_{32}l_{43} + N_1l_{42})y^* \ ,$$

$$N_4' = \frac{dN_4}{dc_5}\Big|_{c_5 = c_5^\star} = (N_1l_{32}l_{43} + p_5l_{42} + l_{31}l_{12}l_{43})y^* \ .$$

Then we are calculate:

$$\overline{\Theta}(c_5^\star)\,\overline{\Psi}(c_5^\star) + \overline{\Gamma}(c_5^\star)\,\overline{\Phi}(c_5^\star) = \left(l_{43}(N_1l_{32} + l_{31}l_{12}) + p_5l_{42} + -\frac{l_{42}N_3}{N_1}\right)(-2N_3y^*) + \frac{2N_3}{N_1^2}(\Delta_1 - N_3)(l_{32}l_{43} - l_{11}l_{42})y^* \neq 0$$

provided that conditions $(4.1a)$ *and* $(4.29) - (4.32)$ (4.34) and (3.6a) are hold. So, we obtain that the Hopf bifurcation occurs around the equilibrium point E_6 at the parameter $c_5 = c_5^\star$ and the proof is complete. ∎

Numerical Simulation of system (2.2) [4]

In this section, we confirmed our obtained results in the previous sections numerically by using Runge Kutta method along with predictor corrector method. Note that, we use turbo C++ in programming and matlab in plotting and then discuss our obtained results. The system (2.2) is studied numerically for different sets of parameters and different sets of initial points. The purpose of studying numerical simulations is to first check for existence of the bifurcation near equilibrium points and secondly confirm our obtained analytical results. It is observed that, for the following set of hypothetical parameters, system (2.2) has a globally asymptotically stable positive equilibrium point as shown in:

Fig.(1) [4].: $$\left.\begin{array}{c} c_1 = 0.5\,, c_2 = 0.4\,, c_3 = 0.4\,, c_4 = 0.5\,, c_5 = 0.3\,, c_6 = 0.01\,, c_7 = 0.1 \\ c_8 = 0.3\,, c_9 = 0.4\,, c_{10} = 0.5, c_{11} = 0.01\,, c_{12} = 0.2, c_{13} = 0.01, c_{14} = 0.1 \end{array}\right\} \qquad (5.1)$$

System (2.2) is solved numerically for the data given in (5.1) with varying one parameter at each time which results the following outputs that represent the numerical bifurcation of system (2.2):

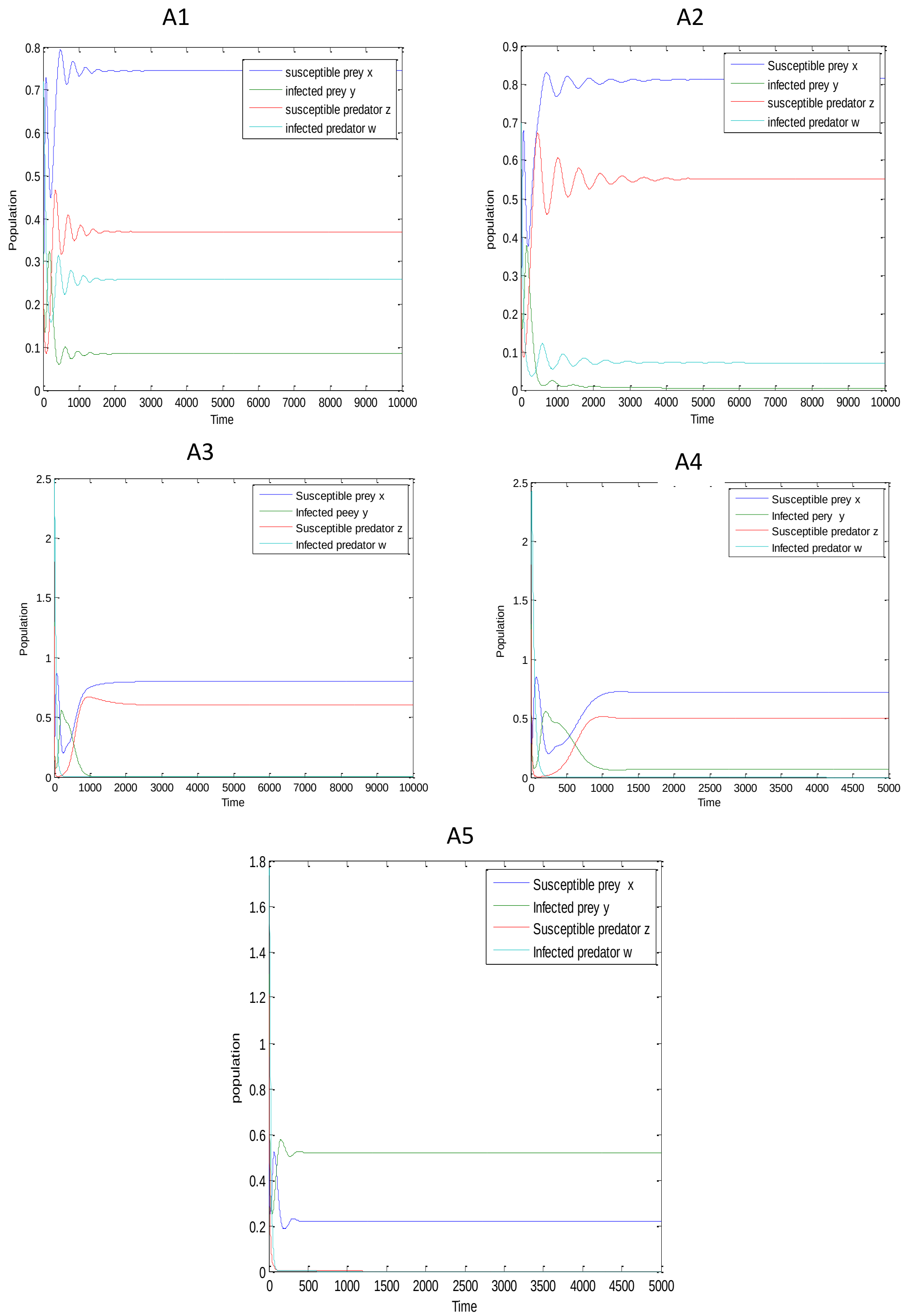

FIGURE 1: Time series of the solution of system (2.2) for the data given in (5.1) with different value of c_{11} : (A1) globally asymptotically stable of the positive equilibrium point $E_6 = (0.75, 0.09, 0.56, 0.27)$ for $c_{11} = 0.167$, (A2) globally asymptotically stable of the infected prey free equilibrium point $E_5 = (0.819, 0, 0.541, 0.094)$ for $c_{11} = 0.169$, while (A3) globally asymptotically stable of the disease free equilibrium point $E_3 = (0.8, 0, 0.6, 0)$ for $c_{11} = 0.2$, (A4) globally asymptotically stable of the infected predator free equilibrium point $E_4 = (0.719, 0.067, 0.499, 0)$ for $c_{11} = 0.25$, (A5) globally asymptotically stable predator free equilibrium point $E_2 = (0.22, 0.52, 0, 0)$ for $c_{11} = 0.6$.
Clearly, figure (1) shows that system (2.2) has a bifurcation at death rate of predator (c_{11}) in the range above keeping other parameters as data given in (5.1)

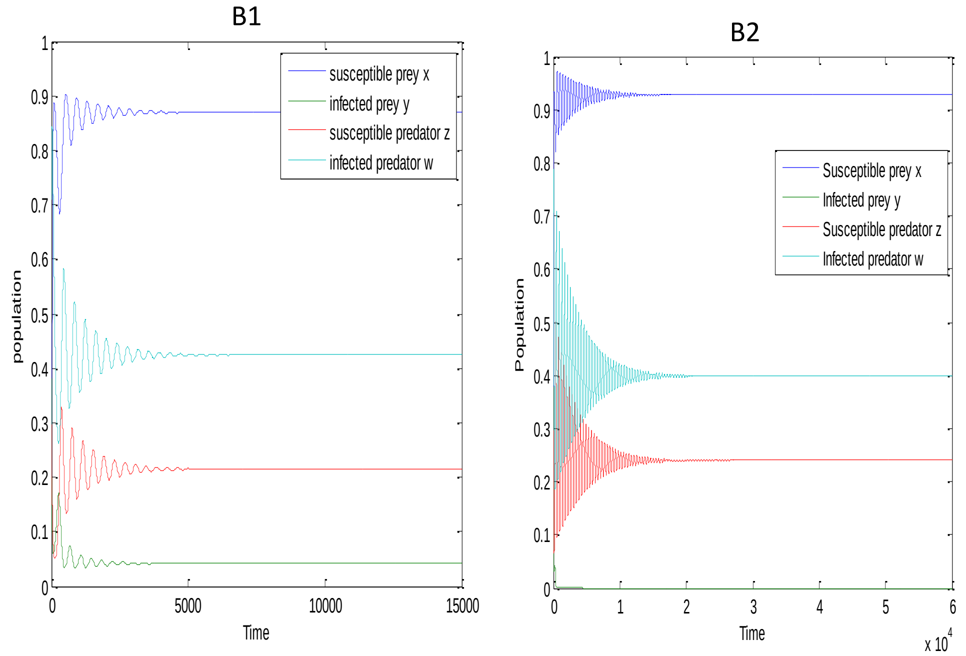

FIGURE 2: (B1) Time series of the solution of system (2.2) approaches to the positive equilibrium point E_6 at $c_6 = 0.123$, while (B2) the time series of the trajectory is approaches asymptotically to the infected prey free equilibrium point $E_5 = (0.92, 0, 0.24, 0.39)$ for the data given in (5.1) with c_6 =0.125,

Clearly, figure (2) shows that system (2.2) has a bifurcation at the death rate of infected prey due to disease rate $c_6 = 0.124$ keeping other parameters as data given in (5.1) .

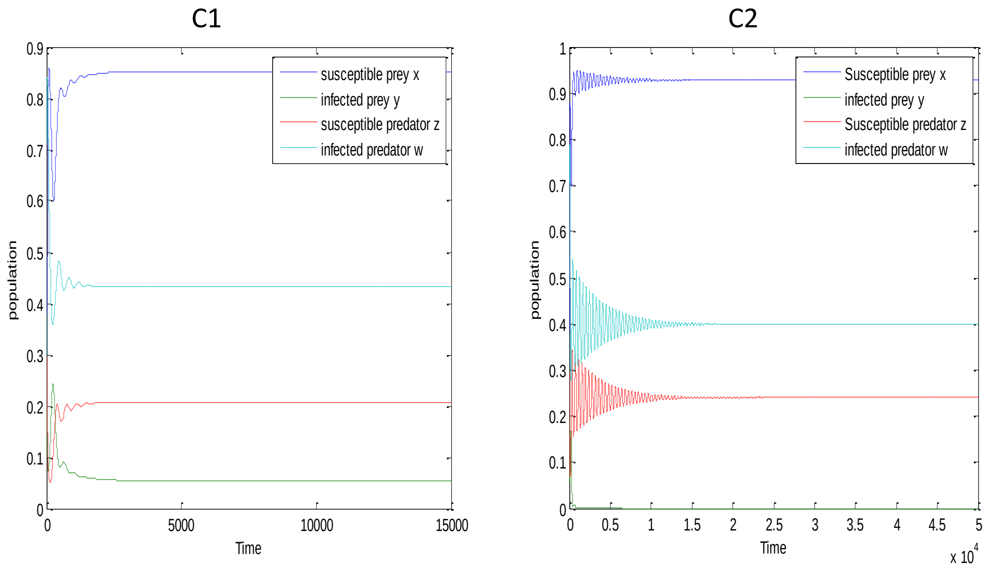

FIGURE 3 :(C1) the time series of the trajectory id approaches to the positive equilibrium point E_6 at $c_4 = 0.96$, while (C2) time series of the solution of system (2.2) approaches asymptotically to the infected prey free equilibrium point $E_5 = (0.92, 0, 0.24, 0.39)$ for the data given in (5.1) with c_4 =0.98.

Clear, figure (3) shows that system (2.2) has a bifurcation at the maximum attack rate for infected prey $c_4 = 0.97$ keeping other parameters as data given in (5.1) .

Moreover system (2.2) is solved numerically for the data given in (5.1) with varying one parameter at each time and the obtained results are given in table (1), for more details see [4].

TABLE **1:** numerical behaviors of system (2.2) for the data given in (5.1) with varying one parameter at each time

Range of parameter	Numerical behavior of system (2.2)	Bifurcation
$0 < c_1 \leq 0.37$	Approach to the infected prey free equilibrium point E_5	
$0.37 < c_1 < 1.5$	Approaches to the positive equilibrium point E_6	
$0.3 < c_2 < 1.45$	Approaches to the positive equilibrium point E_6	
$1.45 \leq c_2$	Approach to the infected prey free equilibrium point E_5	
$0 < c_3 < 1.5$	Approaches to the positive equilibrium point E_6	-
$0.4 < c_4 < 0.97$	Approaches to the positive equilibrium point E_6	
$0.98 < c_4 < 1.5$	Approach to the infected prey free equilibrium point E_5	
$0.2 < c_5 < 0.58$	Approaches to the positive equilibrium point E_6	
$0.58 \leq c_5 < 1.5$	Approach to the infected prey free equilibrium point E_5	
$0 < c_6 \leq 0.124$	Approaches to the positive equilibrium point E_6	
$0.124 < c_6 < 1$	Approach to the infected prey free equilibrium point E_5	
$0 < c_7 < 0.214$	Approaches to the positive equilibrium point E_6	
$0.215 < c_7 < 1$	Approach to the infected prey free equilibrium point E_5	
$0 < c_8 < 0.4$	Approaches to the positive equilibrium point E_6	
$c_8 < 0.012$ and $c_1 < 0.1$	Approaches to the axial equilibrium point E_1	
$0 < c_9 < 0.5$	Approaches to the positive equilibrium point E_6	-
$0.1 < c_{10} < 0.35$	Approach to the infected prey free equilibrium point E_5	
$0.35 < c_{10} < 0.95$	Approaches to the positive equilibrium point E_6	
$0 < c_{11} < 0.168$	Approaches to the positive equilibrium point E_6	
$0.168 \leq c_{11} < 0.2$	Approach to the infected prey free equilibrium point E_5	
$c_{11} = 0.2$	Approach to the disease free equilibrium point E_3	
$0.2 < c_{11} \leq 0.31$	Approach to the infected predator free equilibrium point E_4	
$0.31 < c_{11} \leq 1$	Approaches to the predator free equilibrium point E_2	
$0 < c_{12} \leq 0.3$	Approaches to the positive equilibrium point E_6	-
$0 < c_{13} < 0.09$	Approaches to the positive equilibrium point E_6	
$0.1 \leq c_{13} \leq 0.41$	Approach to the infected prey free equilibrium point E_5	
$0.04 < c_{14} < 0.188$	Approaches to the positive equilibrium point E_6	
$0.188 \leq c_{14} \leq 0.5$	Approach to the infected prey free equilibrium point E_5	

DISCUSSION

In this paper, we studied the conditions of the occurrence of local bifurcation for example (saddle-node, transcritical and pitchfork) with particular emphasis on the Hopf bifurcation near of the positive equilibrium point of eco-epidemiological by using the Sotomayrs theory and the Hopf bifurcation theory.

Mathematical model involving SI infectious disease with harvest in infected population whereas ,this disease cannot transmitted from the prey to the predator or converse , but the disease is transmitted in the same species by contact .The dynamical behavior of system (2.2) has been investigated local bifurcation as well as Hopf bifurcation. Further, system (2.2) has been solved numerically for different sets of initial points and different sets of parameters starting with the hypothetical set of data given by eq. (5.1) and the following observations are obtained.

- The system within the set of parameters imposed does not have a periodic solution.
- The parameters c_3, c_8, c_9 and c_{12} which represent the half saturation, the conversion rate of the susceptible and infected predator c_8 ,c_9 and c_{12} respectively did not play an important role in the bifurcation analysis.
- As increasing the infection rate of prey and predator in the range $c_1 > 0.37$ and $0.35 < c_{10} < 0.95$ respectively and keeping the rest of parameters as in eq. (5.1), the solution of system (2.2) approaches to positive equilibrium point E_6. However if $0.37 \leq c_1 < 1.5$ and $0.1 < c_{10} \leq 0.35$ then the infected prey will face extinction then the trajectory transferred from positive equilibrium point to the equilibrium point E_5, thus, the $c_1 = 0.37$ and $c_{10} = 0.35$ parameter are a bifurcation points**.**
- As, increasing the maximum attack rate of susceptible predator for susceptible and infected prey in the range $0.3 < c_2 < 1.45$ and $0.4 < c_4 < 0.97$ respectively and keeping the rest of parameters as in eq. (5.1), the solution of system (2.2) approaches to positive equilibrium point E_6. However if $1.45 \leq c_2$ and $0.98 < c_4$ then the infected prey will face extinction then the trajectory transferred from positive equilibrium point to the equilibrium point E_5, thus, the $c_2 = 1.45$ and $c_4 = 0.97$ parameters are a bifurcation points**.**
- As increasing the maximum attack rate of infected predator for infected prey , harvesting rate and death of infected predator are due to disease, the parameter in the range $0.2 < c_5 \leq 0.58$,$0 < c_7 \leq 0.214$, $0.04 < c_{14} < 0.188$ and $0 < c_{13} < 0.09$ respectively and keeping the rest of parameters as in eq. (5.1), the solution of system (2.2) approaches to positive equilibrium point E_6. However if $0.58 < c_5$, $0.215 < c_7$,$0.188 \leq c_{14}$ and $0.1 \leq c_{13} \leq 0.041$ then the infected prey will face extinction then the trajectory transferred from positive equilibrium point to the equilibrium point E_5, thus, the $c_5 = 0.58$, $c_7 = 0.214$,$c_{14} = 0.188$ and $c_{13} = 0.1$ parameters are a bifurcation points**.**
- As increasing the death rate of the infected prey due to disease in the range$0 < c_6 < 0.124$ keeping the rest of parameters as in eq. (5.1), the solution of system (2.2) approaches to positive equilibrium point E_6. However if $0.124 < c_6 < 1$ then the infected prey will face extinction then the trajectory transferred from positive equilibrium point to the equilibrium point E_5, thus, the $c_6 = 0.124$
- As the natural death rate of predator c_{11} decrease to 0.168 keeping the rest of parameters as in eq.(5.1), the solution of system (2.2) approaches to positive equilibrium E_6, for more increasing in the range $0.168 \leq c_{11} < 0.2$ causes extinction in the infected prey and the system will approach the infected prey free equilibrium point E_5, further for $c_{11} = 0.2$ the solution of the system (2.2) approaches to the disease free equilibrium point E_3 ;additional for $0.2 < c_{11} \leq 0.31$ causes extinction in the infected predator and the system will approach the infected predator free equilibrium pointE_4 , then more increasing of this parameter in the range $0.31 < c_{11} \leq 1$ the solution of the system (2.2) approaches to the predator free equilibrium point E_2 thus, the c_{11} parameter when $c_{11} = 0.168$, $c_{11} = 0.2$ and $c_{11} = 0.31$ is a bifurcation point.

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